for $1 \le n \le 5$ and by induction $(n+1)(1+(n+2)(n+3)) < 3^n$ for $n \ge 6$, contradicting the fact that $(x, y, n) \in S$.

6. Let the sequence $x_1, x_2, x_3, \ldots$, be defined by $x_1 = a$, where a is a real number, and the recursion $x_{n+1} = 3x_n^3 - 7x_n^2 + 5x_n$ for $n \ge 1$.

Find all values of a for which the sequence has a finite limit as n tends to infinity, and find this limit.

Solved by Arkady Alt, San Jose, CA, USA; Michel Bataille, Rouen, France; and Daniel Tsai, student, Taipei American School, Taipei, Taiwan. We give Bataille's write-up.

Let $f(x) = 3x^3 - 7x^2 + 5x$ and $g(x) = f(x) - x = x(x-1)(3x-4)$. The sequence $\{x_n\}$, which satisfies $x_{n+1} = f(x_n)$ for all positive integers n, can only converge to a root of $g(x) = 0$. Thus, the only possible finite limits of $\{x_n\}$ are 0, 1, and $\frac{4}{3}$. We show that the sequence is convergent if and only if $0 \le a \le \frac{4}{3}$, in which case the limit is 1 except if $a = 0$ and $\lim\limits_{n\to\infty} x_n = 0$ or if $a = \frac{4}{3}$ and $\lim\limits_{n\to\infty} x_n = \frac{4}{3}$.

Suppose first $a < 0$. Since $g(x) < 0$ when $x < 0$, it follows that $x_n < x_1 = a < 0$ for all positive integers n. If $\{x_n\}$ had a finite limit, ℓ, we would have $\ell \le a$, contradicting the fact that $\ell \in \{0, 1, \frac{4}{3}\}$. Thus, $\{x_n\}$ is divergent when $a < 0$. Using the fact that $g(x) > 0$ for $x > \frac{4}{3}$, similar reasoning shows that $\{x_n\}$ is divergent when $a > \frac{4}{3}$.

If $a \in \{0, 1, \frac{4}{3}\}$, then the sequence $\{x_n\}$ is constant.

If $a \in (1, \frac{4}{3})$, then using $f(x) - 1 = (x-1)^2(3x-1)$ an easy induction shows that $1 < x_{n+1} < x_n$ for all positive integers n. Thus, $\{x_n\}$ is decreasing and bounded, hence convergent. Its limit ℓ satisfies $\ell \ge 1$ and $\ell \in \{0, 1, \frac{4}{3}\}$, that is, $\ell = 1$.

If $a \in [\frac{1}{3}, 1)$ then $x_2 = f(a) \ge 1$ and $x_2 < \frac{4}{3}$, as the maximum of f on $[0,1]$ is $f(\frac{5}{9}) = \frac{275}{243} < \frac{4}{3}$. From the previous case, we see that $\lim\limits_{n\to\infty} x_n = 1$.

It remains to study the case $a \in \left(0, \frac{1}{3}\right)$. Then, $\dfrac{1}{3^{m+1}} \le a < \dfrac{1}{3^m}$ for some unique positive integer m. If any of the numbers $x_2, x_3, \ldots, x_m$ is not less than $\frac{1}{3}$, let x_k be the one with the smallest index. Then $\frac{1}{3} \le x_k < \frac{4}{3}$ and by the previous cases $\{x_n\}_{n\ge k}$ converges to 1 and $\lim\limits_{n\to\infty} x_n = 1$. Otherwise, noting that $f(x) - 3x = x(x-2)(3x-1)$ is positive for $x \in \left(0, \frac{1}{3}\right)$, we have

$$\begin{aligned} x_2 &= f(x_1) > 3x_1 = 3a \ge \frac{1}{3^m}, \\ x_3 &= f(x_2) > 3x_2 \ge \frac{1}{3^{m-1}}, \\ &\cdots \\ x_m &= f(x_{m-1}) > 3x_{m-1} \ge \frac{1}{3^2}, \end{aligned}$$

and finally $x_{m+1} > \frac{1}{3}$. So $\{x_n\}_{n\ge m+1}$ converges to 1 and again $\lim\limits_{n\to\infty} x_n = 1$.